$\mathcal{W}(2,4)$, Linear and Non-local $\mathcal{W}$-Algebras in Sp(4) Particle Model

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Abstract

We comment on relations between the linear $\mathcal{W}_{2,4}^{\rm linear}$ algebra and non-linear $\mathcal{W}(2,4)$ algebra appearing in a Sp(4) particle mechanics model by using Lax equations. The appearance of the non-local $V_{2,2}$ algebra is also studied.

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1 Introduction

$\mathcal{W}$-algebras are extensions of the Virasoro algebra [1] with extra bosonic generators having weights in general greater than 2. Recently there appeared two interesting generalizations of the concept of $\mathcal{W}$-algebra.

Linear $\mathcal{W}$-algebras have been introduced by Krivonos, Sorin and Bellucci in [2]. They have shown that, for certain cases, it is possible to embed a $\mathcal{W}$-algebra into a greater non-linear algebra that becomes linear after a redefinition of its generators. Several examples are presented explicitly. For example, by considering a linear algebra spanned by three generators having weights 1, 2 and 4, a new weight 4 generator is constructed by means of a non-linear redefinition of the variables in such a way that the non-linear $\mathcal{W}(2,4)$ algebra becomes a subalgebra. Here the additional weight 1 field plays a crucial rôle. Although $\mathcal{W}(2,4)$ is a subalgebra, this weight 1 field cannot be decoupled from it by a gauge-fixing.

The other generalization is the non-local $V_{n,m}$ algebras developed by Bilal [3]. They are matrix generalizations of $\mathcal{W}$-algebras which arise in the context of non-abelian Toda field theories. The $V_{n,m}$ algebra is the non-linear symmetry of a m-th order differential equation whose coefficients are $n \times n$ matrices.

In previous papers we have examined a set of particle mechanics models with Sp($2M$) Yang-Mills type gauge symmetry [4]. These models present various classical $\mathcal{W}$-symmetries after performing a gauge-fixing induced by an sl(2) embedding in sp($2M$).

In this paper we will point out that both the $\mathcal{W}_{2,4}^{\rm linear}$ algebra and the non-local $V_{2,2}$ algebra are realized at the classical level as subalgebras of the $\mathcal{W}$-algebra obtained from the Sp(4) model after performing the gauge-fixing induced by the sl(2) embedding in sp(4) with characteristic (0,1) [4]. We will refer to this algebra as $\mathcal{W}_{(0,1)}^{\rm sp(4)}$. This algebra has four generators: H, V_2, G and C. The first three generators form a linear subalgebra which is equivalent to the $\mathcal{W}_{2,4}^{\rm linear}$ examined in [2]. In contrast to ref. [2], the weight 1 generator H can be decoupled from the $\mathcal{W}(2,4)$ algebra due to the presence of the additional weight 0 generator C. The resulting algebra is equivalent to one obtained by imposing gauge-fixing conditions on H and C.

In $\mathcal{W}_{(0,1)}^{\rm sp(4)}$ the reduction of the weight 1 generator ($H = 0$) being C free leads to the non-local algebra $V_{2,2}$. The non-locality arises from the inverse of the derivative operator appearing in the definition of the Dirac bracket. We show it more explicitly by constructing starred variables. They have non-local forms by the definition. The further reduction of C recovers the locality and the resulting algebra is $\mathcal{W}(2,4)$ again.

In the next section we give a brief introduction to the Sp(4) model and display the classical algebra of the $\mathcal{W}$-generators. In section 3 the linear $\mathcal{W}$-algebra is discussed in the model. The non-local $\mathcal{W}$-algebra $V_{2,2}$ is derived in section 4. A short summary and

discussions are in the final section. A brief sketch of the relations between the algebras is showed in Fig.1.

$$
\begin{tikzcd}
 & \begin{array}{c}\mathcal{W}^{\mathrm{sp}(4)}_{(0,1)} \\ (H, V_2, G, C)\end{array} \arrow[dl, "H \approx 0"'] \arrow[dr, "C \approx a_0"] & & (H, V_2, G\,;\, C) \arrow[dd] \\
(H) \oplus (V_2^\star, G^\star, C^\star) \quad V_{2,2} \arrow[dr, "C \approx a_0"'] & & \mathcal{W}^{\mathrm{linear}}_{2,4} \arrow[dl, "H \approx 0"] & \\
 & \begin{array}{c}\mathcal{W}(2,4) \\ (H, C) \,\oplus\, (W_2, W_4)\end{array} & & (H, V_2, V_4\,;\, C)
\end{tikzcd}
$$

Fig. 1

2 Particle mechanics model with Sp(4) symmetry

In previous papers [4] we have considered a set of models invariant under a Yang-Mills type gauge symmetry Sp($2M$). They are reparametrization invariant models of M relativistic particles living in a Minkowskian d-dimensional space-time. For the purpose of the present paper we give a brief introduction of the Sp(4) case.

The canonical action is given by:

$$S = \int \mathrm{d}\tau \, \frac{1}{2} \, \bar{R} \, \mathcal{D} \, R, \tag{2.1}$$

where R contains the particle phase space variables and $\bar{R}$ is its conjugate:

$$R = \begin{pmatrix} x_1 \\ x_2 \\ p_1 \\ p_2 \end{pmatrix}, \qquad \bar{R} = (\, p_1, \, p_2, -x_1, -x_2 \,) \, . \tag{2.2}$$

$\mathcal{D}$ is the covariant derivative with respect to the gauge group Sp(4):

$$\mathcal{D} = \frac{\mathrm{d}}{\mathrm{d}\tau} - \Lambda, \qquad \Lambda = \begin{pmatrix} B & A \\ -F & -B^\top \end{pmatrix} \in \mathrm{sp}(4). \tag{2.3}$$

The gauge field Λ is a 4×4 symplectic matrix thus the components A and F are 2×2 symmetric matrices.

In this formulation the gauge invariance of the action is expressed in a manifestly covariant form by means of Yang-Mills type transformations:

$$\delta R = \beta R, \tag{2.4}$$

$$\delta\Lambda = \dot{\beta} - [\Lambda, \beta], \qquad \beta = \begin{pmatrix} \beta_B & \beta_A \\ -\beta_F & -\beta_B^{\top} \end{pmatrix} \in \mathrm{sp}(4). \tag{2.5}$$

where the gauge parameter β is a 4×4 symplectic matrix. The equations of motion of the matter fields are

$$\mathcal{D}R = \dot{R} - \Lambda R = 0. \tag{2.6}$$

The infinitesimal transformation law (2.5) for the gauge variable Λ is the compatibility condition of the pair of equations (2.4) and (2.6) for the matter variable R:

$$0 \;=\; [\,(\delta - \beta),\, \mathcal{D}\,]\, R \;=\; -(\delta\Lambda - \dot{\beta} + [\Lambda, \beta])\, R,$$

and it can be regarded as a zero-curvature condition. The presence of a zero-curvature condition allows us to apply the soldering procedure to reduce the original symmetry of the model to a chiral classical $\mathcal{W}$-symmetry by means of a partial gauge-fixing of the Λ fields. For sp(4) we have three different classes of sl(2) embeddings which will lead to three different gauge-fixings. For the purpose of the present paper we will examine one of these embeddings, namely the sl(2) embedding with characteristic $(0,1)$ [4]. This embedding induces a gauge-fixing which leaves four remnant gauge fields: H, T, G and C. After the gauge-fixing the gauge field Λ is given by:

$$\Lambda_r = \begin{pmatrix} H & 0 & 0 & 1 \\ 0 & -H & 1 & 0 \\ C & \frac{T}{2} & -H & 0 \\ \frac{T}{2} & G & 0 & H \end{pmatrix}. \tag{2.7}$$

In this gauge the action (2.1) becomes, after integrating over the momenta,

$$S = \int \mathrm{d}\tau \left[(\dot{x}_1 - Hx_1)(\dot{x}_2 + Hx_2) + \frac{1}{2}\left(Cx_1^2 + Tx_1x_2 + Gx_2^2 \right) \right]. \tag{2.8}$$

The equations of motion for the matter variables become

$$\begin{pmatrix} C & -(\frac{\mathrm{d}}{\mathrm{d}\tau} + H)^2 + \frac{1}{2}T \\ -(\frac{\mathrm{d}}{\mathrm{d}\tau} - H)^2 + \frac{1}{2}T & G \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0. \tag{2.9}$$

They can be regarded as the Drinfel'd-Sokolov equations for this embedding. The corresponding Lax operator is given in a 2×2 matrix form.

This system has four residual gauge symmetries associated with the four remnant gauge fields. They are:

$\bullet\epsilon$-transformation (Diffeomorphisms):

$$\delta H = \epsilon\dot{H} + H\dot{\epsilon} + \tfrac{k}{2}\ddot{\epsilon}, \qquad \delta T = \epsilon\dot{T} + 2\dot{\epsilon}T - \dddot{\epsilon},$$
$$\delta C = \epsilon\dot{C} + (2-k)C\dot{\epsilon}, \qquad \delta G = \epsilon\dot{G} + (2+k)G\dot{\epsilon},$$
$$\delta x_1 = \epsilon\dot{x}_1 + \tfrac{1}{2}(k-1)x_1\dot{\epsilon}, \qquad \delta x_2 = \epsilon\dot{x}_2 - \tfrac{1}{2}(k+1)x_2\dot{\epsilon},$$

where k is an arbitrary constant coming from the fact that H is an element of Cartan subalgebra. The matter variables x_1 and x_2 transform as primary fields under diffeomorphisms with weights $\frac{1}{2}(k-1)$ and $-\frac{1}{2}(k+1)$ respectively. The gauge variables C and G transform also as primary fields with weights $2-k$ and $2+k$. On the other hand T is a quasi-primary field with weight 2 and H transforms as a field of weight 1 with a $\ddot{\epsilon}$ term.

$\bullet\alpha$-transformation (Dilatations):

$$\delta H = \frac{1}{2}\dot{\alpha}, \qquad \delta T = 0, \qquad \delta C = -\alpha C, \qquad \delta G = \alpha G,$$

$$\delta x_1 = \frac{1}{2}\alpha x_1, \qquad \delta x_2 = -\frac{1}{2}\alpha x_2, \tag{2.10}$$

$\bullet\beta_2$-transformation:

$$\delta H = \frac{1}{2}C\beta_2, \quad \delta T = \beta_2(\dot{C} - 2CH) + 2\dot{\beta}_2 C, \quad \delta C = 0,$$

$$\delta G = \beta_2(4H^3 - 2HT - 6H\dot{H} + \frac{1}{2}\dot{T} + \ddot{H}) - \dot{\beta}_2(6H^2 - T - 3\dot{H}) + 3H\ddot{\beta}_2 - \frac{1}{2}\dddot{\beta}_2,$$

$$\delta x_1 = \beta_2(2Hx_2 + \dot{x}_2) - \frac{1}{2}x_2\dot{\beta}_2 \qquad \delta x_2 = 0, \tag{2.11}$$

$\bullet\beta_5$-transformation: The β_5 transformations can be obtained from the β_2 transformations by the following replacements:

$$\beta_2 \leftrightarrow \beta_5, \qquad H \leftrightarrow -H, \qquad C \leftrightarrow G, \qquad x_1 \leftrightarrow x_2. \tag{2.12}$$

The generators of these transformations are H for the dilatations, C and G for the β_2 and β_5 transformations, respectively, and

$$V_2 \,=\, -\,T \,-\, 2\,H^2 \,+\, 2k\,\dot{H} \tag{2.13}$$

for the diffeomorphisms. The Poisson brackets of these generators are[1]:

$$\{\,H(\tau),\, H(\tau')\,\} \,=\, -\tfrac{1}{4}\,\dot{\delta}, \tag{2.14}$$

[1]All fields on the right hand side of the Poisson brackets depend on τ, the dot means derivative with respect to τ and $\delta \equiv \delta(\tau - \tau')$

$$\{\, H(\tau),\, V_2(\tau')\,\} \;=\; H\,\dot\delta \,+\, \dot H\,\delta \,+\, \tfrac{k}{2}\,\dddot\delta, \tag{2.15}$$
$$\{\, H(\tau),\, G(\tau')\,\} \;=\; \tfrac{1}{2}\,G\,\delta, \tag{2.16}$$
$$\{\, H(\tau),\, C(\tau')\,\} \;=\; -\,\tfrac{1}{2}\,C\,\delta, \tag{2.17}$$
$$\{\, V_2(\tau),\, V_2(\tau')\,\} \;=\; 2\,V_2\,\dot\delta \,+\, \dot V_2\,\delta \,+\, (1+k^2)\,\dddot\delta, \tag{2.18}$$
$$\{\, G(\tau),\, V_2(\tau')\,\} \;=\; (2+k)\,G\,\dot\delta \,+\, \dot G\,\delta, \tag{2.19}$$
$$\{\, C(\tau),\, V_2(\tau')\,\} \;=\; (2-k)\,C\,\dot\delta \,+\, \dot C\,\delta, \tag{2.20}$$
$$\{\, C(\tau),\, C(\tau')\,\} \;=\; \{\, G(\tau),\, G(\tau')\,\} \;=\; 0 \tag{2.21}$$

and

$$\begin{aligned}\{\, G(\tau),\, C(\tau')\,\} \;&=\; \left(-8\,H^3 \,-\, 2\,H\,V_2 \,+\, (4k+8)\,H\,\dot H \,-\, (1+k)\,\ddot H \,+\, \frac{1}{2}\,\dot V_2\right)\,\delta \,+ \\ &+\,\left(8\,H^2 \,-\, (3+2k)\,\dot H \,+\, V_2\right)\,\dot\delta \,-\, 3\,H\,\ddot\delta \,+\, \frac{1}{2}\,\dddot\delta.\end{aligned} \tag{2.22}$$

The Poisson brackets (PB) (2.14)–(2.22) between the generators H, V_2, G and C are linear except for the last one $\{G, C\}$. Furthermore H, V_2 and G form a subalgebra. This linear subalgebra happens to be equivalent (for $k = 2$) to the $\mathcal{W}_{2,4}^{\rm linear}$ algebra of ref. [2].

In the next section we are going to show the relations between $\mathcal{W}_{(0,1)}^{\rm sp(4)}$, $\mathcal{W}_{2,4}^{\rm linear}$ and $\mathcal{W}(2,4)$ from the study of the Lax equations.

3 Linear $\mathcal{W}$-algebra from Lax equation

Let us consider the equations of motion (2.9) for the matter variables or Lax equations for the $\mathcal{W}_{(0,1)}^{\rm sp(4)}$ algebra. Assuming the division by C we can obtain a fourth order equation for x_2,

$$x_2^{(4)} \,+\, u_1\,\dddot x_2 \,+\, u_2\,\ddot x_2 \,+\, u_3\,\dot x_2 \,+\, u_4\,x_2 \;=\; 0. \tag{3.1}$$

The coefficient of the third order derivative term is $u_1 \;=\; -2\frac{\dot C}{C}$. It disappears if the equation is expressed for $y = C^{-1/2}x_2$:

$$y^{(4)} \,+\, W_2\,\ddot y \,+\, \dot W_2\,\dot y \,+\, \left(W_4 \,+\, \frac{9}{100}\,W_2^2 \,+\, \frac{3}{10}\,\ddot W_2\right)\,y \;=0, \tag{3.2}$$

where W_2, W_4 are given by:

$$W_2 \;=\; -\,T \,-\, 2\,h^2 \,+\, 4\,\dot h, \qquad h \;\equiv\; H + \frac{\dot C}{2C} \tag{3.3}$$

and

$$\begin{aligned}W_4 \;&=\; -\,C\,G \,+\, \frac{4}{25}\,W_2^2 \,+\, \frac{1}{5}\,\ddot W_2 \,+\, 2\,h^2\,W_2 \,-\, 2\,\dot h\,W_2 \,-\, h\,\dot W_2 \,+ \\ &+4\,h^4 \,+\, 7\,\dot h^2 \,-\, 16\,h^2\,\dot h \,+\, 6\,h\,\ddot h \,-\, \dddot h.\end{aligned} \tag{3.4}$$

Note that h is a field having zero PB with H. The Poisson brackets between W_2 and W_4 close giving the classical $\mathcal{W}(2,4)$ algebra:

$$\{ W_2(\tau),\, W_2(\tau') \} \;=\; 2\, W_2\, \dot{\delta} \,+\, \dot{W}_2\, \delta \,+\, 5\, \dddot{\delta}, \tag{3.5}$$

$$\{ W_4(\tau),\, W_2(\tau') \} \;=\; 4\, W_4\, \dot{\delta} \,+\, \dot{W}_4\, \delta, \tag{3.6}$$

$$\begin{aligned} \{ W_4(\tau),\, W_4(\tau') \} \;=\; & \tfrac{1}{20}\, \delta^{(7)} \,+\, \tfrac{7}{25}\, W_2\, \delta^{(5)} \,+\, \tfrac{7}{10}\, \dot{W}_2\, \delta^{(4)} \,+ \\ & + \left(\tfrac{21}{25}\, \ddot{W}_2 \,+\, \tfrac{49}{125}\, W_2^2 \,+\, \tfrac{3}{5}\, W_4 \right) \dddot{\delta} \,+\, \left(\tfrac{14}{25}\, \dddot{W}_2 \,+\, \tfrac{147}{125}\, W_2\, \dot{W}_2 \,+\, \tfrac{9}{10}\, \dot{W}_4 \right) \ddot{\delta} \,+ \\ & + \left(\tfrac{1}{5}\, W_2^{(4)} \,+\, \tfrac{88}{125}\, W_2\, \ddot{W}_2 \,+\, \tfrac{59}{100}\, \dot{W}_2^2 \,+\, \tfrac{72}{625}\, W_2^3 \,+\, \tfrac{1}{2}\, \ddot{W}_4 \,+\, \tfrac{14}{25}\, W_2\, W_4 \right) \dot{\delta} \,+ \\ & + \left(\tfrac{3}{100}\, W_2^{(5)} \,+\, \tfrac{177}{500}\, \dot{W}_2\, \ddot{W}_2 \,+\, \tfrac{39}{250}\, W_2\, \dddot{W}_2 \,+\, \tfrac{108}{625}\, W_2^2\, \dot{W}_2 \,+\, \tfrac{1}{10}\, \dddot{W}_4 \,+\, \tfrac{7}{25}\, (W_4 W_2)^{\cdot} \right) \delta. \end{aligned} \tag{3.7}$$

Furthermore W_2 and W_4 have zero Poisson bracket both with H and C. This implies that W_2 and W_4 are combinations invariant under α and β_2 transformations. Thus we can gauge fix H and C independently of W_2 and W_4. This decoupling procedure of C and H is equivalent to imposing a set of gauge fixing conditions: $H = 0$ and $C = a_0$, constant. Regarding them as second class constraints and eliminating them we obtain the non-linear $\mathcal{W}(2,4)$ algebra realized by Dirac brackets.

The appearance of the $\mathcal{W}(2,4)$ algebra is a consequence of the transformation of the $\mathcal{W}^{\mathrm{sp}(4)}_{(0,1)}$ Lax equations (2.9) into the $\mathcal{W}(2,4)$ Lax equation (3.2). The procedure described above gives in an easy way the expression of the $\mathcal{W}(2,4)$ generators in terms of the $\mathcal{W}^{\mathrm{sp}(4)}_{(0,1)}$ generators (3.3)–(3.4).

If we further analyze the algebra (H, W_2, W_4, C) by imposing the gauge-fixing condition $C = a_0$, where a_0 is a non-vanishing constant, we will find out the appearance of a linear algebra. Since we are interested in the gauge fixing $C = a_0$, the conformal weight of C should be zero. Therefore we should change the energy momentum tensor. The corresponding energy-momentum tensor is given by V_2 (2.13) with $k = 2$:

$$V_2 \;=\; -\,T \,-\, 2\, H^2 \,+\, 4\, \dot{H}, \tag{3.8}$$

which is obtained from W_2 (3.3) by putting $C = a_0$, *i.e.* by replacing h by H. The new weight 4 generator, V_4, is also given by the same replacement in W_4:

$$\begin{aligned} V_4 \;\;=\;\; & -\, a_0\, G \,+\, \frac{4}{25}\, V_2^2 \,+\, \frac{1}{5}\, \ddot{V}_2 \,+\, 2\, H^2\, V_2 \,-\, 2\, \dot{H}\, V_2 \,-\, H\, \dot{V}_2 \,+ \\ & +\, 4\, H^4 \,+\, 7\, \dot{H}^2 \,-\, 16\, H^2\, \dot{H} \,+\, 6\, H\, \ddot{H} \,-\, \dddot{H}. \end{aligned} \tag{3.9}$$

The fields V_2 and V_4 close under Poisson brackets forming again the non-linear $\mathcal{W}(2,4)$ algebra. The interesting observation is that the generators H, V_2 and G form a linear algebra, which is equivalent to the $\mathcal{W}^{\mathrm{linear}}_{2,4}$ algebra of ref. [2] and, therefore, equation (3.9) gives in a natural way the non-linear change of variables that relates the linear algebra (H, V_2, G) with the non-linear one (V_2, V_4).

The relations between the different algebras can be summarized in this scheme:

$$\begin{array}{ccc} \mathcal{W}_{(0,1)}^{\mathrm{sp}(4)} = (H, V_2, G, C) & \supset & \mathcal{W}_{2,4}^{\mathrm{linear}} = (H, V_2, G) \\ \downarrow & & \downarrow \\ \mathcal{W}(2,4) \oplus (H, C) & & (H, V_2, V_4) \supset \mathcal{W}(2,4) \end{array}$$

The vertical arrows symbolize a non-linear change of variables induced by the Lax equations.

We believe that the study of the Lax equations described above could be useful to construct other classical linear $\mathcal{W}$-algebras and the non-linear changes relating them to their non-linear counterparts.

4 Non-local $\mathcal{W}$-algebra

Non-local extensions of $\mathcal{W}$-algebras were discussed by Bilal [3]. We can see how the simplest one, $V_{2,2}$, arises in our model. It appears after a reduction of H in the $\mathcal{W}_{(0,1)}^{\mathrm{sp}(4)}$ algebra of section 2. This reduction is possible by defining the Dirac bracket or by introducing starred variables. We follow the latter approach since it will make clear the reason why the non-linear algebra appears.

Starred variables have zero PB with second-class constraints ϕ^a. Their primitive form is:

$$A^\star \;=\; A \;-\; \{A, \phi^a\}(\{\phi^a, \phi^b\})^{-1}\, \phi^b, \tag{4.1}$$

which is valid up to linear terms of constraints, *i.e.*, terms containing higher powers of constraints are not taken into account. Here we take $H(\tau)$ as ϕ^a and keep all orders of constraints. $H^\star$ is zero by the definition (4.1). Others are:

$$V_2^\star \;=\; V_2 + 2H^2 - 2k\dot{H} \;=\; -T \tag{4.2}$$

and

$$G^\star(\tau) \;=\; G(\tau)\, e^{-2\int^\tau d\tau' H(\tau')}, \qquad C^\star(\tau) \;=\; C(\tau)\, e^{2\int^\tau d\tau' H(\tau')}. \tag{4.3}$$

They have zero PB with H:

$$\{V_2^\star, H\} \;=\; \{G^\star, H\} \;=\; \{C^\star, H\} \;=\; 0. \tag{4.4}$$

The origin of the non-locality is the inverse of $\dot{\delta}$ in the PB $\{H(\tau), H(\tau')\}$ (2.14).

The three starred generators ($V_2^\star$, $G^\star$ and $C^\star$) span the non-local algebra $V_{2,2}$ discussed by Bilal [3] under Poisson brackets:

$$\{V_2^\star(\tau), V_2^\star(\tau')\} \;=\; 2\, V_2^\star(\tau)\, \dot{\delta} \;+\; \dot{V}_2^\star(\tau)\, \delta \;+\; \dddot{\delta}, \tag{4.5}$$

$$\{C^\star(\tau), V_2^\star(\tau')\} \;=\; 2\, C^\star(\tau)\, \dot{\delta} \;+\; \dot{C}^\star(\tau)\, \delta, \tag{4.6}$$

$$\{G^\star(\tau), V_2^\star(\tau')\} \;=\; 2\, G^\star(\tau)\,\dot\delta \,+\, \dot G^\star(\tau)\,\delta, \tag{4.7}$$

$$\{C^\star(\tau), C^\star(\tau')\} \;=\; \tfrac{1}{2}\, C^\star(\tau)\, C^\star(\tau')\,\epsilon(\tau'-\tau), \tag{4.8}$$

$$\{G^\star(\tau), G^\star(\tau')\} \;=\; \tfrac{1}{2}\, G^\star(\tau)\, G^\star(\tau')\,\epsilon(\tau'-\tau), \tag{4.9}$$

$$\{C^\star(\tau), G^\star(\tau')\} = -\tfrac{1}{2}\, C^\star(\tau)\, G^\star(\tau')\,\epsilon(\tau'-\tau) \,+\, V_2^\star(\tau)\,\dot\delta \,+\, \tfrac{1}{2}\,\dot V_2^\star(\tau)\,\delta + \tfrac{1}{2}\dddot\delta. \tag{4.10}$$

This result is also expected from the fact that the equation of motion of the model (2.9) is a 2×2 matrix second-order Lax equation and the condition $H=0$ guarantees the absence of first order derivative term. $C^\star$ and $G^\star$ correspond to $V^\pm$ of ref. [3]. $C^\star$ and $G^\star$ are real generators while V^+ and V^- appear as mutually conjugate ones. We can further reduce $C^\star$ by constructing generators $V_2^{\star\star}$ and $V_4^{\star\star}$ having zero PB both with H and C. Generators satisfying this requirement have been found and are precisely W_2 in (3.3) and W_4 in (3.4). The non-locality disappeared since $G^\star$ and $C^\star$ appear by their product in W_4. The recovery of locality is understood by the fact that the matrix of constraints,

$$\begin{pmatrix} \{H(\tau),\, H(\tau')\} & \{H(\tau),\, C(\tau')\} \\ \{C(\tau),\, H(\tau')\} & \{C(\tau),\, C(\tau')\} \end{pmatrix} = \begin{pmatrix} -\frac{1}{4}\,\partial_\tau\delta & -\frac{C}{2}\,\delta \\ \frac{C}{2}\,\delta & 0 \end{pmatrix} \tag{4.11}$$

has a local inverse:

$$\begin{pmatrix} 0 & \frac{2}{C}\,\delta \\ -\frac{2}{C}\,\delta & -\frac{1}{C}\,\partial_\tau\left(\frac{1}{C}\,\delta\right) \end{pmatrix}. \tag{4.12}$$

5 Summary

In this paper we have shown that $\mathcal{W}^{\mathrm{sp}(4)}_{(0,1)}$, which is the symmetry algebra of the Sp(4) particle mechanics model gauge-fixed by means of the $(0,1)$ sl(2) embedding, encodes both linear and non-local $\mathcal{W}$-algebras. We reduce the fields H and C one by one. By first reducing H we get the non-local $V_{2,2}$ algebra [3]. On the other hand, the reduction of C first gives a linear $\mathcal{W}$-algebra [2]. Reduction of both H and C ends with $\mathcal{W}(2,4)$ any way.

The imposition of $H=0$ is regarded as a gauge-fixing condition by itself. The non-locality enters from the inverse of the Poisson bracket (2.14) appearing in the definition of Dirac bracket. Furthermore, by imposing $C=a_0$ the non-locality disappears due to the locality of the inverse of the PB matrix (4.11) and the resulting residual algebra is $\mathcal{W}(2,4)$.

On the other hand, reduction of C in $\mathcal{W}^{\mathrm{sp}(4)}_{(0,1)}$ leaves three generators (H, V_2 and G) satisfying a linear algebra which is equivalent to the linear $\mathcal{W}$-algebra examined in ref. [2]. $\mathcal{W}(2,4)$ generators are constructed through a non-linear and invertible transformation (3.9). In contrast to the previous case ($H=0$), the reduction $C=a_0$ is not regarded as a gauge-fixing condition since $\{C,C\}=0$ does not have inverse — it is a first class constraint and is regarded as a restriction on the solutions. The algebra

generated by H, V_2 and G is a subalgebra of the original $\mathcal{W}^{\mathrm{sp}(4)}_{(0,1)}$ algebra but not commuting with C. Finally $\mathcal{W}^{\mathrm{sp}(4)}_{(0,1)}$ is decomposed into two mutually decoupled algebras with generators (H, C) and (W_2, W_4). The latter are given in (3.3) and (3.4) and are generators of $\mathcal{W}(2, 4)$, which is the residual symmetry after reducing both H and C.

We emphasize that the linear algebra and the non-linear change of variables that relates it with the non-linear $\mathcal{W}$-algebra appear naturally from the study of the Lax equations.

The generalization of present discussions to other cases is interesting. By considering Sp(6) models [4] we may discuss the relations between the linear $\mathcal{W}_3^{\mathrm{linear}}$, non-local $V_{3,3}$ and non-linear $\mathcal{W}_3$ and $\mathcal{W}_3^2$ algebras. Finally we add a remark about two papers on linear $\mathcal{W}$-algebras that have appeared recently [5]. Their results coincide with our conclusions when they overlap.

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